THE RELATIVE EMPHASIS UPON MECHANICAL SKILL AND APPLICATIONS OF ELEMENTARY MATHEMATICS

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The teaching of elementary mathematics has two distinct purposes, quite closely connected with each other and yet easily distinguished. The pupil must develop a skill in the manipulation of the symbols of elementary mathematics and he must develop a power to apply them to the solution of life problems. By special emphasis upon either phase of the subject an ability can be developed in it without a corresponding ability in the other phase, and sometimes at the expense of ability in the other phase. It is easy to develop in pupils a high degree of skill in the mechanical processes of computation with almost no ability to decide in a particular problem what process is to be used. And in pioneer days many men were quite able to solve the problems met in daily life though knowing nothing of the use of figures for purposes of computation. It has been very interesting to me to note that pupils in the third and fourth grades, utterly innocent of the use of the symbols in division of fractions, will frequently solve problems involving division of fractions more readily than will those same pupils after they have been taught to "invert the divisor and multiply." This does not mean that skill in computation with figures stands in the way of solving concrete problems, but that during the time devoted to acquiring the skill they have lost power in thinking number relations, possibly through a transfer of interest.

In selecting the content of elementary mathematics shall we be controlled by the purpose of training in ability to solve applied problems or by the purpose of developing a high degree of skill with symbols, or by a combination of the two? If we select the content on the basis of the applications, we should, of course, include only those mechanical processes which are of use in solving problems. In arithmetic, though there are several perfectly good ways of subtracting with figures, if our purpose is purely to give a useful tool to be used in problem solving, we should teach but one of them. If our primary purpose is to

develop skill in the use of symbols in computation, we shall teach him several ways of subtracting and develop skill in each. That we are not agreed on either of these purposes is shown by the fact that both of these plans are being followed in different schools to-day. In algebra if our primary purpose is the ability to solve problems we shall include in factoring only those types which the pupil may need in solving problems. Shall we teach the factors of $a^3 - b^3$? At this moment I can think of no concrete problem whose solution depends either directly or indirectly upon skill in factoring $a^3 - b^3$. If selecting the content of factoring on the basis of its applications, I would then omit this type. If, however, I were selecting the content of factoring on the basis of the mechanical processes I would include this type, for it furnishes a very interesting part of the systematic development of the topic of factoring. Having factored a^2-b^2, it is natural to want to go on to a^n-b^n. In algebra we are constantly facing the question, "Are we teaching a science of algebraic symbols, or are we developing power to use algebraic symbols to solve life problems?" The answer to this question must necessarily be, "We are doing both." But in selecting the content of algebra and arithmetic I shall assume that our *primary* purpose should be the second—to develop power to use symbols in *solving problems.*

If you agree with me in this as the dominating purpose when selecting the content of arithmetic, I believe we must modify the content of arithmetic largely. The change should come not so much in the *kind* of material now given as in the *quantity*. I do not believ that we are teaching mechanical processes not needed in the solution of problems, but I do believe that we are failing to develop an ability to solve problems requiring the degree of skilled secured in computation, and that we are even tending to make problem solving itself mechanical. Pupils are taught to solve problems by types just as they are taught factoring by types. They learn to recognize the process to be performed in a problem, not by thinking the conditions stated, but by recognizing set phrases. Give the problem "One ship can cross the Atlantic in 8 days. How long will it take 3 ships?" and you will probably get the answer "24 days," because the phrasing of the problem suggests multiplication.

As early as the second grade the pupil begins to develop skill in computation with figures. He is also given some experience in thinking number relations. But the greater part of his time and energy is devoted to securing skill with figures and the solving of problems seems to be used merely to give meaning to the mechanical processes. Generally there is not enough experience in problem solving to be successful even in this. It seems to be desirable here to reverse the emphasis entirely. Instead of using problems to illustrate mechanical processes, develop skill in the processes only so far and so fast as it is needed in solving problems. This does not imply that skill in computation is to be developed only through problem solving. Drill is necessary and much of it should be entirely detached from the applications—should be purely manipulation of symbols. But this drill should usually be preceded by applied problems making clear the nature of the process needed and showing the need for skill in it. If this approach through applied problems seems impossible, then certainly after a degree of skill has been secured the processes should be followed rather promptly by applications. To each the manipulation of complicated complex fractions in elementary algebra because the pupil will need that skill in simplifying the results of differentitation in calculus is certainly a waste of energy. I am inclined to make the extent of the mechanical processes entirely dependent upon the need for them in solving the problems which the pupil is soon to be asked to solve. In arithmetic we very properly teach the process of short division after the pupil has had considerable experience with problems in which fractional parts are taken and one group is measured by another. His mastery of this process with larger numbers is followed by problems in which his newly acquired skill is exercised. Our only error here seems to be in developing too little power in problem interpretation and judgment of processes to be used in proportion to the skill developed in computation.

If we are in error at all in the relative emphasis upon processes and applications in arithemetic, it seems to me we are much more so in algebra. In arithmetic we give no processes which are not needed in the applications which follow rather promptly. Our error is purely an error of relative emphasis. But in alge-

bra we give skill in mechanical manipulation of symbols which is not only not needed in the accompanying applications but in some cases is not needed in any future applications. And in most cases the processes are taught before the problems in which they are to be used are faced or understood by the pupil at all.

The introduction to algebraic notation will illustrate this point. In arithmetic, after the pupil has had considerable experience with five objects of various kinds he is given the figure 5, which then has some content for him. On the first page of his algebra he is usually told that in algebra numbers are also represented by letters. The distinction between the symbol a and the symbol 2 as representing numbers can have no meaning for him. He should first have some general number experiences, then be given the general number symbol. Let him see that a rectangle 2″ x 3″ contains 2x3 sq. in., then enough similiar cases to conclude that the area of any rectangle is equal to the product of its base and its altitude, and he is soon ready for a symbol which represents the base of *any* rectangle. Through many applications dealing with general number and the building up of many formulas the pupil will have built up a symbolization full of meaning, and will have some knowledge of how to use it from the very applications from which it was derived. In using these general number symbols to generalize and formulate the rules of arithmetic he will necessarily learn how to indicate the fundamental processes. For example, if he works out the rule for the area of a trapezoid and attempts to state it as a formula, he will see that he must have a way of indicating that b and b' are to be added, the result multiplied by h and the product divided by 2. Having had such experiences as this in using symbols to indicate processes which a problem requires him to perform, he will later have little difficulty in learning the order of operations and the use of parenthesis in expressions to be evaluated or interpreted. This approach to the symbolization and mechanical processes of algebra through problem conditions differs widely in both plan and results from the more usual plan of first introducing the symbols, defining their use, and then using examples and, finally, problems to illustrate their meaning and use. The first plan begins with problems, goes to mechanical processes as tools for solving them, and ends with problems in which the

new tool is needed. The second plan begins with symbols and and mechanical processes, goes to problems which illustrate these processes, and ends with mechanical processes for which these problems have prepared. By the first plan algebra is a tool for use in life situations; by the second, a science of number symbols illustrated by concrete conditions. I know, of course, that algebra is essentially both a science of symbols and a tool applied to problem solving, but in selecting its content I believ that the latter conception of it should control, and that only those things should be taught in algebra which will be needed rather promptly to solve problems which may possibly be met with in later life.

This should modify our teaching of algebra in two ways. It would compel us to extend the period of exposure to elementary algebra and to omit many of the processes which we are now teaching. To build up a content for the algebraic notation through formulating the rules of arithmetic it becomes advisable to introduce the new notation soon after the rules have been made by the pupils. This could certainly be begun to advantage as early as the seventh grade, probably a year earlier. In my judgment this early introduction is an advantage to the work in arithmetic as well as an introduction to the symbols of general number.

I feel very confident that if the content of elementary algebra were based entirely on the applications to probable life problems, many eliminations of topics would result, possibly a few new ones would be introduced, and certainly the relative emphasis upon topics would be changed. A great part of the space now devoted to the mechanical processes is given to long division, factoring, complicated fractions, theory of exponents, and radicals. I very much doubt whether any of the texts give problems which involve long division either directly or indirectly. The emphasis upon factoring is out of all proportion to its use in solving problems and many types are given which are not involved directly or indirectly in the solution of the book problems. Types of fractions are taught which the pupil meets in no problems earlier than the calculus, and the same can be said of the work with exponents. By reducing largely the amount of time devoted to these mechanical processes, the processes retained could be given a richer meaning through more abundant applications.

Do the students now completing elementary algebra have real power in the use of algebraic symbols, or have they merely developed skill in recognizing types? We teach a pupil to develop the quadratic formula—that is, we develop it for him and then say "Do you see?" and after a while he does. Suppose we then say to him, "Develop a formula for the linear equation in one unknown." Can he do it? Can he even take the first step and write a general equation in one unknown? He can repeat glibly that "the product of the sum and the difference of two numbers is equal to the difference of their squares," but suppose you ask him to show you that the sum of the sum and the difference of two numbers is equal to twice the greater number, will he use general number symbols and do it? Or give him the little number trick, "Think any number, multiply it by 6, add 24, divide by 3, add 4, divide by 2, subtract the original number. The result is 6." Ask him to show how it works. Can he discover? And yet it is in just these ways of formulating number laws and interpreting general number conditions that the symbolization of algebra is useful. We have not been teaching him to make formulas and state laws, but only to solve a very limited number of type problems. Later in life the problems he meets do not come under the types he was taught to solve, he was not given enough experience in problem solving to develop power in applying the algebraic notation to the problems he does meet, and he joins the ranks of those who insist that algebra is not useful except as mental discipline. It seems to me that we have done enough and probably too much toward developing skill in the mechanical processes of algebra, almost enough toward skill in solving problems whose conditions are definitely set up and according to type, but far too little toward developing power in using the algebraic notation to formulate number laws and set up problem conditions. And I am very confident that this change in emphasis would rather increase than diminish the pupil's ability to master the more advanced mathematics if he reaches it.

In the discussion of algebra I have included under the head of applications both the concrete problem situations to which the processes are to be applied, and the method of thought by which the application to to be made. Using algebraic symbols

to state a number law or a rule I have classed under the applications of algebra. It is in essence the method of algebraic thought. I have endeavored to show that in algebra it is this method of thought that we have failed to teach among the applications. In geometry the same division into mechanical processes, method of thought, and concrete applications exists but the present relative emphasis upon them is quite different from that in algebra. The mechanical processes in geometry are very meager, soon mastered, and therefore not over-emphasized. We mechanize the order of a proof, the definitions, and the theorems in order. But these are necessarily repeated so often that after the first few weeks little emphasis is placed upon the mechanical processes. The contest for emphasis is now between the method of thought in geometry and the applications. When teaching the geometry of similiar triangles, which should we stress most, the method of proof of the theorems or their application to concrete problems? Originally the proofs of theorems were the only applications considered as belonging to geometry. The application of the facts of geometry to life problems was merely mensuration and was degraded to a mere topic of arithmetic. Later it became customary to introduce an increasing number of applied problems immediately after the proof of a theorem, though the proof of the theorem and the problem were kept quite distinct. Recently there has been a marked tendency to introduce as early as the seventh grade the notions and the names practically all of the geometric figures, to teach the theorems of geometry without proof, and to use these facts in a great variety of applied problems. The pupil becomes familiar with the language of geometry. He is led to discover the facts and learns to make accurate geometric drawings. He collects and remembers these facts as a means to solving numerous problems of construction and mensuration. Problems of this kind are found much more within the range of his interests and his abilities than are the problems of business usually given in the seventh grade.

The plan of having inventional, or problem solving geometery, precede demonstrative geometry has several advantages. In algebra the power to use algebraic symbols in formulating general number laws is so directly connected with ability to solve con-

crete problems that each is directly helpful to the other and they should be trained together. In geometry, however, the ability to prove a theorem has little if any connection with ability to understand or to apply it, so that either can be taught without the other. I have sometimes felt that the attempt to combine the two was a detriment to both. The type of mind which is interested in working out the logical demonstration of a theorem is not apt at that time to be interested in using that theorem to measure the height of a tree. And if one is inerested in finding the height of a tree and knows the theorem by which to do it, he is apt to care little for the proof of the theorem. I am very skeptical of the theory that demonstrative geometry can be motivated by applied problems. I much prefer to motivate demonstrative geometry by an easy original exercise than to use a problem in mensuration. If the applications are the interesting things, why have the proofs, is a very natural question, especially in view of the fact that the proofs throw little or no light upon the applications. I believe thoroughly that the method of geometric proof should be retained, probably as a requirement for all pupils, because of its influence upon all our thinking. But I would have a course in geometry preliminary to demonstrative geometry. This course, distributed through the seventh and eighth grades, would include notions of all the geometric forms, skill in constructing them, the theorems about them, and problems applying these facts. This would of course, include the elementary facts of trigonometry along with other facts of similar triangles. Following this I would give a course in demonstrative geometry whose purpose would be the mastery of the method of geometric proof through a well selected sequence of easily proved theorems. Since pupil motive in any school activity is in the success he achieves in it, rather than in any realization of future need for it, I should hope to so present the early theorems and their proofs that the pupils could master them. In that case he will be ready for the next proof, and would regard an applied problem as an obstacle rather than as an inducement to the next demonstration.

By way of summary, then, permit me to say that in arithmetic it seems to me that we have selected the content as we should on the basis of the applications. But I think that we are over-

stressing the mechanical processes in relation to the applications, and are presenting them too early—before the child is prepared for them through the applications. In algebra we have not selected the content on the basis of the applications, as we should, but rather on the basis of the mechanical processes. Much of the work upon the mechanical processes should be eliminated. Much more emphasis should be placed upon the applications, especially the algebraic method of using general numbers. The work in algebra, especially its applications, should be distributed over a longer period. In geometry I believe that the stress is where it should be, upon the applications, but that the applied problems of geometry may well precede and be entirely separated from the training in the geometric method of proof.